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COMMENT

Lie symmetries and invariants for the time-dependent generalizations of the equation $\ddot{\boldsymbol{R}} + C_1 R^n \boldsymbol{L} + C_2 R^m \boldsymbol{R} = 0$

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Abstract. In this comment we use an invertible point transformation for the study of time-dependent equations derived from the equation $\ddot{\boldsymbol{R}} + C_1 R^n \boldsymbol{L} + C_2 R^m \boldsymbol{R} = 0$. This procedure generalizes some results obtained recently by Leach and Gorringe.

1. Introduction

In a recent paper, published in this journal, Leach and Gorringe (1990) analysed the Lie symmetries for the equation

$$\ddot{\boldsymbol{R}} + h(R)\boldsymbol{L} + q(R)\boldsymbol{R} = 0 \tag{1}$$

where $\boldsymbol{L} = \boldsymbol{R} \times \dot{\boldsymbol{R}}$ and $\ddot{\boldsymbol{R}} = \mathrm{d}^2\boldsymbol{R}/\mathrm{d}T^2$. Two special cases of this equation are particularly important from a physical point of view: the Kepler problem, where $h(R) = 0$ and $q(R) = kR^{-3}$, and the charge-monopole problem, if $h(R)) = CR^{-3}$ and $q(R) = 0$. In this comment we give some time-dependent generalizations of these systems; they are obtained by using invertible point transformations. We find the general expression for the system of differential equations which are equivalent to the equation

$$\ddot{\boldsymbol{R}} + C_1 R^n \boldsymbol{L} + C_2 R^m \boldsymbol{R} = 0. \tag{2}$$

under a special point transformation. Equation (2) is sufficiently general for our purpose: it is a particular form of equation (1) of Leach and Gorringe (1990). All the properties of the transformed system, its symmetry generators, invariants and solutions can be found from the knowledge of the corresponding properties of equation (2). From this analysis several of the results by Katzin and Levine (1983) and by Leach and Gorringe (1990) are directly found and generalized. We discuss also the Lie symmetry structure for the generalized equation of motion of a charged particle in the field of a magnetic dipole.

The Lie algebra associated with the symmetry generators of equation (1), in the general case, is $\mathrm{a}_1 \oplus \mathrm{SO}(3)$, with a_1 representing the symmetry under time translation and SO(3) the rotational invariance. The symmetry generators, in this case, are

$$U_1 = \frac{\partial}{\partial T} \qquad U_2 = Z\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial Z}$$
$$U_3 = X\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial X} \qquad U_4 = Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}. \tag{3}$$

If we make the invertible point transformation

$$\boldsymbol{R}=f(t)\boldsymbol{r} \qquad T=g(t) \tag{4}$$

the equation (2) will be transformed to the equation

$$\ddot{\boldsymbol{r}}+f_1(t)\dot{\boldsymbol{r}}+f_2(t)\boldsymbol{r}+C_1f_3(t)r^n\boldsymbol{l}+C_2f_4(t)r^m\boldsymbol{r}=0 \tag{5}$$

where $\boldsymbol{l}=\boldsymbol{r}\times\dot{\boldsymbol{r}}$ and

$$\begin{aligned} &2\dot{f}/f-\ddot{g}/\dot{g}=f_1 \qquad f^{n+1}\dot{g}=f_3 \\ &\ddot{f}/f-(\ddot{g}/\dot{g})(\dot{f}/f)=f_2 \qquad f^m\dot{g}^2=f_4. \end{aligned} \tag{6}$$

The system (6) can be solved in terms of f_1 and f_3, if $n\neq-3$,

$$\begin{aligned} f&=f_3^{1/(n+3)}\exp\left(\frac{1}{(n+3)}\int f_1\,\mathrm{d}t\right) \\ g&=\int f_3^{2/(n+3)}\exp\left(-\frac{(n+1)}{(n+3)}\int f_1\,\mathrm{d}t\right)\mathrm{d}t \\ f_4&=f_3^{(m+4)/(n+3)}\exp\left(\frac{(m-2n-2)}{(n+3)}\int f_1\,\mathrm{d}t\right) \end{aligned} \tag{7}$$

and f_2 satisfies

$$f_2=-\frac{(n+4)}{(n+3)^2}\dot{f}_3^2/f_3^2+\frac{1}{(n+3)}\ddot{f}_3/f_3+\frac{(n+1)}{(n+3)^2}f_1\dot{f}_3/f_3+\frac{1}{(n+3)}\dot{f}_1+\frac{(n+2)}{(n+3)^2}f_1^2. \tag{8}$$

If we impose $f_1=0$ in (7) we get

$$f=\dot{g}^{1/2}. \tag{9}$$

Making $f=W^{-1}(t)$ the equation (5) becomes

$$\ddot{\boldsymbol{r}}-(\ddot{W}/W)\boldsymbol{r}+C_1W^{-(n+3)}r^n\boldsymbol{l}+C_2W^{-(m+4)}r^m\boldsymbol{r}=0. \tag{10}$$

For the case $n=-3$ the solution of equations (6) is

$$\begin{aligned} &f_1=-\dot{f}_3/f_3 \qquad f=f_4^{1/(m+4)}f_3^{-2/(m+4)} \\ &f_2=\ddot{f}/f-(\dot{f}_3/f_3+2\dot{f}/f)\dot{f}/f \qquad g=\int f_3f^2\,\mathrm{d}t. \end{aligned} \tag{11}$$

We observe that the reverse of our procedure was discussed in a paper by Berkovich and Rozov (1972) where they applied a method of transforming non-autonomous nonlinear equations into autonomous ones.

2. Equations with a scale symmetry

We will find firstly the most general equation (2) with a scale symmetry. If we impose this symmetry

$$U_5=T\frac{\partial}{\partial T}+aX^i\frac{\partial}{\partial X^i} \tag{12}$$

and use the Lie conditions for the equation (2), we find the following equation:

$$\ddot{\boldsymbol{R}}+C_1R^n\boldsymbol{L}+C_2R^{2(n+1)}\boldsymbol{R}=0. \tag{13}$$

The motivation for the choice of the vector field U_5 (and of the vector field U_6 below) was the analysis made by Leach and Gorringe (1990) where similar systems were considered.

The equation (13) has the symmetry generators U_1, U_2, U_3, U_4 and the additional symmetry generator

$$U_5 = T\frac{\partial}{\partial T} - (n+1)^{-1} X^i \frac{\partial}{\partial X^i}. \tag{14}$$

The Lie algebra associated with these generators is $a_2 \oplus SO(3)$.

The time-dependent equation, with the same Lie symmetry structure but with transformed symmetry generators, obtained from (13) and (4), is

$$\ddot{\boldsymbol{r}} + f_1 \dot{\boldsymbol{r}} + f_2 \boldsymbol{r} + C_1 f_3 r^n \boldsymbol{l} + C_2 f_4 r^{2(n+1)} \boldsymbol{r} = 0. \tag{15}$$

The f_i are given by (7), if $n \neq -3$, and by (11) if $n = -3$.

If we search for a particular case of equation (2) with the additional symmetry

$$U_6 = T^2 \frac{\partial}{\partial T} + T X^i \frac{\partial}{\partial X^i} \tag{16}$$

the Lie conditions lead to the equation

$$\ddot{\boldsymbol{R}} + C_1 R^{-3} \boldsymbol{L} + C_2 R^{-4} \boldsymbol{R} = 0. \tag{17}$$

The Lie algebra associated with the symmetry generators $U_1, \ldots, U_6$ is, in this case, $sl(2, R) \oplus SO(3)$. The transformed equation is

$$\ddot{\boldsymbol{r}} + f_1 \dot{\boldsymbol{r}} + f_2 \boldsymbol{r} + C_1 f_3 r^{-3} \boldsymbol{l} + C_2 f_4 r^{-4} \boldsymbol{r} = 0 \tag{18}$$

where the f_i are given by (6).

3. Time-dependent case of the Kepler problem considered by Katzin and Levine (1983)

Let $m = -3$, $C_1 = 0$ in (2). Solving (6), with $f_1 = 0$, we get

$$\begin{aligned} &f_3 = C_0 \qquad f = f_4 \\ &f_2 = \ddot{f}_4/f_4 - 2\dot{f}_4^2/f_4^2. \end{aligned} \tag{19}$$

Making $f_4 = W^{-1}(t)$, equation (5) is reduced to the equation

$$\ddot{\boldsymbol{r}} - (\ddot{W}/W)\boldsymbol{r} + kW^{-1} r^{-3} \boldsymbol{r} = 0 \tag{20}$$

under the point transformation

$$\boldsymbol{R} = W^{-1} \boldsymbol{r} \qquad T = \int W^{-2}\, \mathrm{d}t. \tag{21}$$

The symmetry generators for the Kepler problem are U_1, U_2, U_3, and U_4 in (3), and

$$U_5 = T\frac{\partial}{\partial T} + \frac{2}{3} X^i \frac{\partial}{\partial X^i}. \tag{22}$$

The symmetry generators for (20), obtained by Katzin and Levine (1983), can be determined directly from (3), (21) and (22). By using the transformation (21) the conserved Laplace–Runge–Lenz vector can be generalized for equation (20). It takes the form

$$\boldsymbol{I}_1 = \boldsymbol{l} \times (W\dot{\boldsymbol{r}} - \dot{W}\boldsymbol{r}) + kr^{-1}\boldsymbol{r}. \tag{23}$$

4. Time-dependent generalization of the charge-monopole and charge-dipole problems

The equation describing the pure charge-monopole interaction is a particular case of equation (2); if we choose $C_2=0$, $n=-3$ it leads to

$$\ddot{\boldsymbol{R}}+C_1R^{-3}\boldsymbol{L}=0. \tag{24}$$

Solving (6) for this case and choosing $f_3=1$, $f=W^{-1}$, we get, from (5), the equation

$$\ddot{\boldsymbol{r}}-(\ddot{W}/W)\boldsymbol{r}+C_2r^{-3}\boldsymbol{l}=0 \tag{25}$$

which describes a charge-monopole interaction plus a time-dependent linear force. This case can be generalized by including the force $C_2R^{-4}\boldsymbol{R}$. In this case equation (25) becomes

$$\ddot{\boldsymbol{r}}-(\ddot{W}/W)\boldsymbol{r}+C_1r^{-3}\boldsymbol{l}+C_2r^{-4}\boldsymbol{r}=0 \tag{26}$$

with the same Lie symmetry group as the charge-monopole equation (24): sl(2, R)⊕ SO(3) (Moreira *et al* 1985).

The invariants for equation (24) are

$$\begin{aligned} &\boldsymbol{J}=\boldsymbol{L}-C_1R^{-1}\boldsymbol{R} \qquad && E=\dot{R}^2/2 \\ &I_1=\dot{\boldsymbol{R}}\cdot(\boldsymbol{R}-\dot{\boldsymbol{R}}T) && I_2=(\boldsymbol{R}-\dot{\boldsymbol{R}}T)^2. \end{aligned} \tag{27}$$

From them the invariants for the transformed equation (25) can be found easily. They are

$$\begin{aligned} &\boldsymbol{J}=\boldsymbol{l}-C_1r^{-1}\boldsymbol{r} \qquad E=(W\dot{\boldsymbol{r}}-\dot{W}\boldsymbol{r})^2/2 \\ &I_1=\boldsymbol{r}\cdot\dot{\boldsymbol{r}}(1+2W\dot{W}D)-W^2D\dot{r}^2-\dot{W}(W^{-1}-\dot{W}D)r^2 \\ &I=[W^{-1}\boldsymbol{r}-(W\dot{\boldsymbol{r}}-\dot{W}\boldsymbol{r})D]^2 \end{aligned} \tag{28}$$

where $D=\int W^{-2}\,\mathrm{d}t$. These time-dependent systems are three-dimensional Ermakov systems with a given symmetry structure.

We consider now another similar problem. The interaction between an electric charge and a magnetic dipole is an important physical problem; in recent years the scattering of the charged particle, in this problem, has been analysed as an example of chaotic scattering (Jung and Scholz 1988). We discuss now the Lie symmetry structure for the equation which describes this motion. The magnetic field produced by a dipole, with magnetic moment $\boldsymbol{M}=A_0\hat{\boldsymbol{k}}$, is

$$\boldsymbol{B}=R^{-5}[3(\boldsymbol{M}\cdot\boldsymbol{R})\boldsymbol{R}-R^2\boldsymbol{M}] \tag{29}$$

and the equation for an electrically charged parrticle in this field is given by

$$\ddot{\boldsymbol{R}}+\lambda R^{-5}[\dot{\boldsymbol{R}}\times(R^2\hat{\boldsymbol{k}}-3(\hat{\boldsymbol{k}}\cdot\boldsymbol{R})\boldsymbol{R}] \tag{30}$$

where $\lambda=qA_0/mc$, and the vector $\boldsymbol{M}$ is directed along the Z axis.

By applying the twice extended operator U'' to the equation (30) we arrive at the following solutions for the symmetry generators:

$$\begin{aligned} &U_1=\frac{\partial}{\partial T} \qquad U_2=X\frac{\partial}{\partial Y}-Y\frac{\partial}{\partial X} \\ &U_3=T\frac{\partial}{\partial T}+(X/3)\frac{\partial}{\partial X}+(Y/3)\frac{\partial}{\partial Y}+(Z/3)\frac{\partial}{\partial Z}. \end{aligned} \tag{31}$$

The physical meaning of these symmetries is clear. The vector field U_1 represents the time translation invariance, U_2 the invariance under rotations around the Z-axis and U_3 a scale symmetry. The commutation relations are

$$[U_1, U_2]=0 \qquad [U_1, U_3]=U_1 \qquad [U_2, U_3]=0 \tag{32}$$

corresponding to the Lie algebra $a_1 \oplus a_2$.

The application of the point transformation (4) to equation (30) leads to

$$\ddot{\boldsymbol{r}}+f_1(t)\dot{\boldsymbol{r}}+f_2(t)\boldsymbol{r}+\lambda f_3(t)r^{-5}\dot{\boldsymbol{r}}\times(r^2\hat{\boldsymbol{k}}-3z\boldsymbol{r})+\lambda f_4(t)r^{-1}\boldsymbol{r}\times\hat{\boldsymbol{k}}=0 \tag{33}$$

where

$$\begin{aligned} &f_1=2\dot{f}/f-\ddot{g}/\dot{g} \qquad f_2=\ddot{f}/f-\ddot{g}\dot{f}/\dot{g}f \\ &f_3=\dot{g}/f^3 \qquad f_4=\dot{g}\dot{f}/f^4. \end{aligned} \tag{34}$$

These equations can be solved in terms of f_1 and f_3, and we get

$$\begin{aligned} &f=f_3^{-1}\exp\left(-\int f_1\,\mathrm{d}t\right) \qquad g=\int f_3^{-2}\exp\left(-3\int f_1\,\mathrm{d}t\right)\mathrm{d}t \\ &f_4=-(\dot{f}_3+f_1f_3) \qquad f_2=-(\ddot{f}_3/f_3+3f_1\dot{f}_3/f_3+\dot{f}_1+2f_1^2). \end{aligned} \tag{35}$$

Making $f=W^{-1}(t)$, equation (33) becomes

$$\ddot{\boldsymbol{r}}-(\ddot{W}/W)\boldsymbol{r}+\lambda Wr^{-5}\dot{\boldsymbol{r}}\times(r^2\hat{\boldsymbol{k}}-3z\boldsymbol{r})-\lambda\dot{W}r^{-3}\boldsymbol{r}\times\hat{\boldsymbol{k}}=0. \tag{36}$$

The following invariants for the equation (30) can be easily determined:

$$I_1=\dot{R}^2/2;\ I_2=(X\dot{Y}-\dot{X}Y)+\lambda(X^2+Y^2)R^{-3}$$

and the corresponding invariants for (33) are given by

$$\begin{aligned} &I_1=[(f/\dot{g})\dot{\boldsymbol{r}}+(\dot{f}/g)\boldsymbol{r}]^2/2 \\ &I_2=(f^2/\dot{g})(x\dot{y}-\dot{x}y)+\lambda f^{-1}r^{-3}(x^2+y^2). \end{aligned}$$

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